Problem 1.23

A man stands atop a mountain whose altitude is given by $z = e^{-(x^4+4y^2)}$ and pours boiling oil upon the climbers below him. What paths do the rivulets of oil follow? [Assume that these paths are orthogonal to the contour lines (level curves) of the mountain.]

Solution

We can see what this mountain looks like by plotting z as a function of x and y.

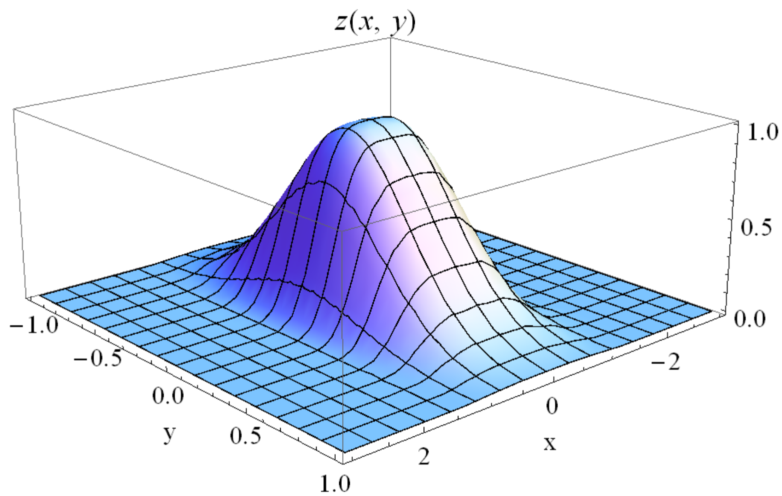

Figure 1: Plot of $z(x, y)$ for $-3 < x < 3$ and $-1 < y < 1$ and $0 < z < 1$.

A level curve of the mountain is a set of points that all have the same height on the mountain. Since the height is represented by z, the general expression for the level curves is obtained by setting z equal to a constant C.

$$e^{-(x^4+4y^2)} = C$$

This equation tells us the set of points on the mountain at every possible height in the xy-plane. Since the paths the rivulets of oil follow are orthogonal to the contour lines, we will have to solve this equation for the orthogonal trajectories.

$$\begin{aligned} \ln e^{-(x^4+4y^2)} &= \ln C \\ -(x^4+4y^2) &= \ln C \\ x^4+4y^2 &= A \end{aligned}$$

Now differentiate both sides with respect to x.

$$\begin{aligned} 4x^3 + 8yy' &= 0 \\ x^3 + 2yy' &= 0 \\ 2yy' &= -x^3 \\ y' &= -\frac{x^3}{2y} \end{aligned}$$

The orthogonal trajectories have slopes given by the negative reciprocal.

$$y_T' = \frac{2y_T}{x^3}$$

Solve for y_T by separation of variables.

$$\frac{dy_T}{y_T} = 2x^{-3}\,dx$$

Integrate both sides.

$$\ln|y_T| = -x^{-2} + C_1$$
$$|y_T| = e^{-x^{-2}+C_1}$$
$$y_T = \pm e^{C_1} e^{-x^{-2}}$$

Therefore, the orthogonal trajectories are represented by the following family of curves. The rivulets of oil follow these trajectories.

$$y_T(x) = Be^{-\frac{1}{x^2}}$$

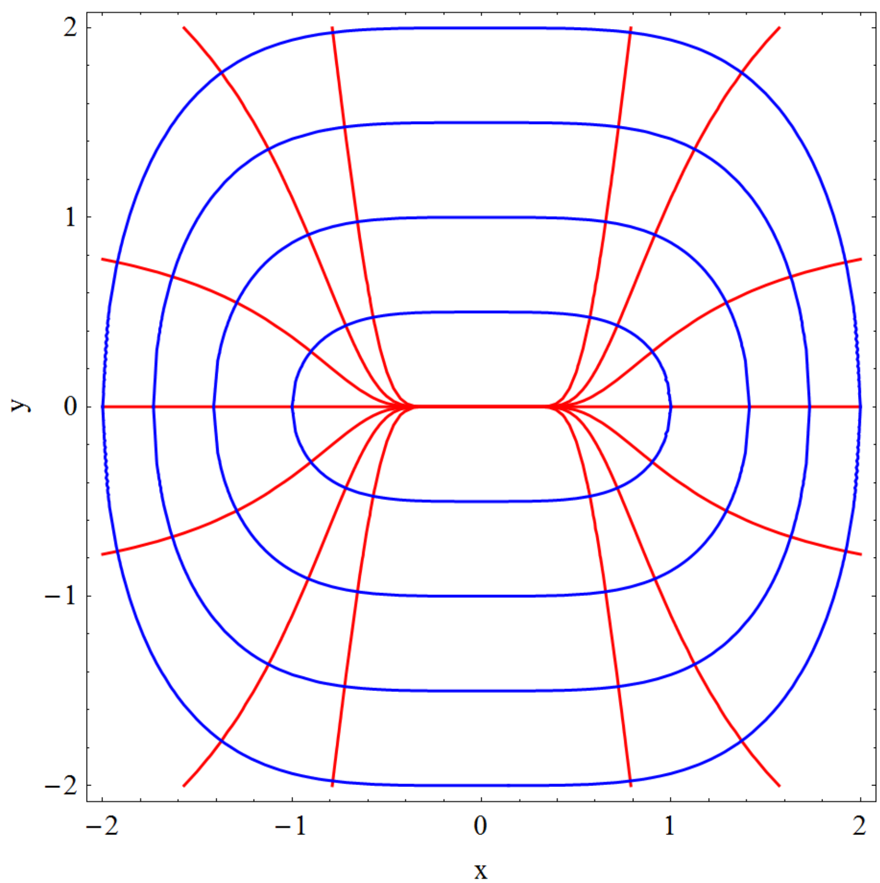

Figure 2: Plot of some of the level curves of z in blue ($A = \{1, 4, 9, 16\}$) and some of the paths the rivulets of oil follow in red ($B = \{0, \pm1, \pm3, \pm10\}$).